

# DEAD ENDS IN MISÈRE PLAY: THE MISÈRE MONOID OF CANONICAL NUMBERS

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## Abstract

We find the misère monoids of normal-play canonical-form integer and non-integer numbers. These come as consequences of more general results for the universe of *dead-ending* games. Left and right *ends* have previously been defined as games in which Left or Right, respectively, have no moves; here we define a dead left (right) end to be a left (right) end whose options are all left (right) ends, and we define a dead-ending game to be one in which all end followers are dead. We find the monoids and partial orders of dead ends, integers, and all numbers, and construct an infinite family of games that are equivalent to zero in the dead-ending universe.

## Keywords

Combinatorial game; Partizan; Misère; Monoids; Dead-ending.

## 1 Introduction

In many *combinatorial games* (two-player games of perfect information and no chance), players take turns placing pieces on a board according to some set of rules. Usually these rules imply that the board spaces available to a player on his or her turn are a subset of those available on the previous turn; games

such as DOMINEERING, COL, SNORT, HEX, and NOGO, among many others, fit this description. What sorts of properties to these *placement games* have over other games? What can be said about their game trees? In contrast to a game like MAZE or KONANE, placement games have the property that a player cannot ‘open up’ moves for him or herself, or for the opponent; in particular, if a player has no available moves at some position of the game then they will have no moves in any follower<sup>1</sup> of that position. This particular property, which we call *dead-ending*, is the focus of the present paper.

A *left end* is a game position with no options for the player we call Left, and a *right end* is a position with no options for the player Right. A game with no options for either player (called the *zero game*) is both a left end and a right end. We can thus define *dead-ending* games as follows.

**Definition 1.** *A left (right) end is a dead end if every follower is also a left (right) end. A game  $G$  is called dead-ending if every end follower of  $G$  is a dead end. The set of all dead-ending games is denoted  $\mathcal{E}$ .*

In addition to the games listed above, many well-studied positions from normal-play game theory have the dead-ending property: integers are dead ends, and non-integer numbers, all-small games, and all HACKENBUSH positions are dead-ending. The set of all dead-ending games is thus a meaningful (and large) universe to consider. Restricted universes are of particular interest to those studying *misère* games (where the last player to move loses), since the restrictions may reintroduce some of the algebraic structure that is lost in general *misère* play. The dead-ending universe is a natural extension of the dicot<sup>2</sup> universe, which has been the focus of recent research in *misère* game theory (see [2],[3], and [5], for example).

In this paper we establish some basic results for dead-ending games and demonstrate that several significant subuniverses (ends, integers, and non-integer numbers) have many of the ‘nice’ algebraic properties that are missing from general *misère* play. More specifically, we find the *misère monoids* of these sets of games, and determine the associated partial orders. The concept of a *misère monoid*, and other prerequisite material, is discussed in Section 1.1.

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<sup>1</sup>By *follower* we mean any position that can be reached from a given game position, by alternating or non-alternating play, including the original position itself.

<sup>2</sup>In a *dicot* game (called *all-small* normal play), the position and every follower satisfies the property that either both players have a move or the game is over. These games are trivially dead-ending, as no follower can be a non-zero end.

## 1.1 General misère background

By convention, the players Left and Right are female and male, respectively. Under *normal play*, the first player unable to move in a game loses; the less-studied and less-structured ending condition known as *misère play* declares that the first player unable to move is the winner. Games or *positions* are defined in terms of their options:  $G = \{G^L \mid G^R\}$ , where  $G^L$  is the set of positions  $G^L$  to which Left can move in one turn, and similarly for  $G^R$ . The simplest game is the zero game,  $0 = \{\cdot \mid \cdot\}$ , where the dot indicates an empty set of options.

In both play conventions, the outcome classes *next* ( $\mathcal{N}$ ), *previous* ( $\mathcal{P}$ ), *left* ( $\mathcal{L}$ ), and *right* ( $\mathcal{R}$ ) are partially ordered as shown in Figure 1, with Left preferring moves toward the top and Right preferring moves toward the bottom. We use  $\mathcal{N}^-$ ,  $\mathcal{P}^-$ ,  $\mathcal{L}^-$ , and  $\mathcal{R}^-$  to denote the outcome classes under misère play. We also use the outcome functions  $o^-(G)$  and  $o^+(G)$  to distinguish between the misère and normal outcomes, respectively.

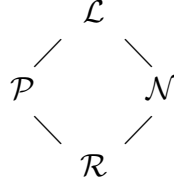


Figure 1: The partial order of outcome classes.

Many definitions from normal-play game theory<sup>3</sup> are used without modification for misère games, including birthday, disjunctive sum, equality, and inequality. Thus, for misère games,

$$G = H \text{ if } o^-(G + X) = o^-(H + X) \text{ for all games } X,$$

$$G \geq H \text{ if } o^-(G + X) \geq o^-(H + X) \text{ for all games } X.$$

In normal-play, the *negative* of a game is defined recursively as  $-G = \{-G^R \mid -G^L\}$ , and is so-called because  $G + (-G) = 0$  for all games  $G$ . Under general misère play, however, this property holds only if  $G$  is the zero game [? ]. To avoid confusion and inappropriate cancellation, we write  $\overline{G}$  instead of  $-G$  and refer to this game as the *conjugate* of  $G$ .

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<sup>3</sup>A complete overview of normal-play game theory can be found in [1].

In normal-play games, there is an easy test for equality:  $G = 0$  if and only if  $o^+(G) = \mathcal{P}$ , and so  $G = H$  if and only if  $o^+(G - H) = \mathcal{P}$ . In misère-play, no such test exists. Equality of misère games is difficult to prove and is, moreover, relatively rare: for example, besides  $\{\cdot | \cdot\}$  itself, there are no games equal to the zero game under misère play [? ]. As with normal-play, we can reduce a misère game to a unique canonical form by eliminating dominated options and bypassing reversible ones [10], but instances of domination (inequality) and reversibility are much less common under misère play. Plambeck (see [8] and [9], for example) introduced a partial solution to these challenges: modify the definitions of equality and inequality by restricting the game universe. Specifically, given a set of games  $\mathcal{U}$ , *misère equivalence (modulo  $\mathcal{U}$ )* is defined by

$$G \equiv H \pmod{\mathcal{U}} \text{ if } o^-(G + X) = o^-(H + X) \text{ for all games } X \in \mathcal{U},$$

while *misère inequality (modulo  $\mathcal{U}$ )* is defined by

$$G \geq H \pmod{\mathcal{U}} \text{ if } o^-(G + X) \geq o^-(H + X) \text{ for all games } X \in \mathcal{U}.$$

We use the words *equivalent* and *indistinguishable* interchangeably, and if  $G \not\equiv H \pmod{\mathcal{U}}$  then  $G$  and  $H$  are said to be *distinguishable*. If  $G \not\geq H$  and  $G \not\leq H$  then  $G$  and  $H$  are *incomparable*, and we write  $G || H$ . In this paper we use the symbol  $\not\geq$  to indicate strict modular inequality. The set  $\mathcal{U}$  is often called the *universe*.

Given a universe  $\mathcal{U}$ , we can determine the equivalence classes under  $\equiv \pmod{\mathcal{U}}$  and form the quotient semi-group  $\mathcal{U}/\equiv$ . This quotient, together with the tetra-partition of elements into the sets  $\mathcal{P}^-$ ,  $\mathcal{N}^-$ ,  $\mathcal{R}^-$ , and  $\mathcal{L}^-$ , is called the *misère monoid* of the set  $\mathcal{U}$ , denoted  $\mathcal{M}_{\mathcal{U}}$ . It is usually desirable to have the set of games  $\mathcal{U}$  closed under disjunctive sum; when a set of games is not already thus closed, we consider its *closure* or set of all sums of those games.

Plambeck and Siegel [9] used the monoid approach with much success in analyzing impartial misère games. Allen [2, 3] extended the idea to partizan game theory, investigating monoids within the dicot universe. Most recently, McKay, Milley, and Nowakowski [5] determined the monoid for a subset of the dicot universe corresponding to instances of HACKENBUSH, and Milley, Nowakowski, and Ottaway [7] found the monoid for ends in the universe of alternating (or ‘consecutive-move-ban’) games. In this paper we make a significant contribution to the literature by determining the misère monoid

of all normal-play canonical-form numbers. We find this to be the same as the monoid of the closure of dead ends.

In Section 2 we establish some basic properties of the dead-ending universe. In Section 3 we analyze ends in the dead-ending universe, which includes all integers in normal-play canonical form. In Section 4 we extend this analysis to non-integer numbers and find that the monoid of all numbers is equivalent to the monoid of integers. We also determine the partial orders of these subuniverses (modulo the subuniverse as well as modulo  $\mathcal{E}$ ), and establish invertibility of the elements (modulo  $\mathcal{E}$ ). Finally, in Section 5 we discuss other dead-ending games, in the context of equivalency to zero modulo the dead-ending universe.

## 2 Preliminary results

We begin with some immediate consequences of the definition of dead-ending. Lemmas 1 and 2 show that the universe  $\mathcal{E}$  of dead-ending games is ‘closed’ in two important respects: it is closed under followers and closed under disjunctive sum.

**Lemma 1.** *If  $G$  is dead-ending then every follower of  $G$  is dead-ending.*

*Proof.* If  $H$  is a follower of  $G$ , then every follower of  $H$  is also a follower of  $G$ ; thus if  $G$  satisfies the definition of dead-ending, then so does  $H$ .  $\square$

**Lemma 2.** *If  $G$  and  $H$  are dead-ending then  $G + H$  is dead-ending.*

*Proof.* Any follower of  $G + H$  is of the form  $G' + H'$  where  $G'$  and  $H'$  are (not necessarily proper) followers of  $G$  and  $H$ , respectively. If  $G' + H'$  is a left end, then both  $G'$  and  $H'$  are left ends, which must be dead, since  $G$  and  $H$  are dead-ending. Thus, any right options  $G'^R$  and  $H'^R$  are left ends, and so all options  $G'^R + H'$  and  $G' + H'^R$  of  $G' + H'$  are left ends. A symmetric argument holds if  $G' + H'$  is a right end, and so  $G + H$  is dead-ending.  $\square$

Under misère play, Left trivially wins any left end playing first. In general, Left may or may not win a left end playing second; for example, the game  $\{\cdot \mid 1\}$  is a left end in  $\mathcal{N}^-$ . If a (non-zero) left end is dead, however, then it is a win for Left playing first or second.

**Lemma 3.** *If  $G \neq 0$  is a dead left end then  $G \in \mathcal{L}^-$ , and if  $G \neq 0$  is a dead right end then  $G \in \mathcal{R}^-$ .*

*Proof.* A left end is always in  $\mathcal{L}^-$  or  $\mathcal{N}^-$ . If  $G$  is a dead left end then any right option  $G^R$  is also a left end, so Right has no good first move. Similarly, a dead right end is in  $\mathcal{R}^-$ .  $\square$

The following lemma, which describes a sufficiency condition for invertibility, applies generally to any misère universe. In the present paper we apply it to various subsets of dead-ending games.

**Lemma 4.** *Let  $U$  be any game universe closed under conjugation, and let  $S \subseteq U$  be a set of games closed under followers. If  $G + \overline{G} + X \in \mathcal{L}^- \cup \mathcal{N}^-$  for every game  $G \in S$  and every left end  $X \in \mathcal{U}$ , then  $G + \overline{G} \equiv 0 \pmod{\mathcal{U}}$  for every  $G \in S$ .*

*Proof.* Let  $S$  be a set of games with the given conditions. Since  $U$  is closed under conjugation, by symmetry we also have  $G + \overline{G} + X \in \mathcal{R}^- \cup \mathcal{N}^-$  for every  $G \in S$  and every right end  $X \in \mathcal{U}$ .

Let  $G$  be any game in  $S$  and assume inductively that  $H + \overline{H} \equiv 0 \pmod{\mathcal{U}}$  for every follower  $H$  of  $G$ . Let  $K$  be any game in  $\mathcal{U}$ , and suppose Left wins  $K$ . We must show that Left can win  $G + \overline{G} + K$ . Left should follow her usual strategy in  $K$ ; if Right plays in  $G$  or  $\overline{G}$  to, say,  $G^R + \overline{G} + K'$ , with  $K' \in \mathcal{L}^- \cup \mathcal{P}^-$ , then Left copies his move and wins as the second player on  $G^R + \overline{G}^L + K' = G^R + \overline{G^R} + K' \equiv 0 + K'$ , by induction. Otherwise, once Left runs out of moves in  $K$ , say at a left end  $K''$ , she wins playing next on  $G + \overline{G} + K''$  by assumption.  $\square$

In subsequent sections we refer to the two game functions below, which are well-defined for our purposes — namely, for numbers and ends.

**Definition 2.** *The left-length of a game  $G$ , denoted  $l(G)$ , is the minimum number of consecutive left moves required for Left to reach zero in  $G$ . The right-length  $r(G)$  of  $G$  is the minimum number of consecutive right moves required for Right to reach zero in  $G$ .*

In general, left- and right-length are well-defined if  $G$  has a non-alternating path to zero for Left or Right, respectively, and if the shortest of such paths is never dominated by another option. The latter condition ensures  $l(G) = l(G')$  when  $G \equiv G'$ . As suggested above, both of these conditions are met if  $G$  is a (normal-play) canonical-form number or if  $G$  is an end in  $\mathcal{E}$ . If  $l(G)$  and  $l(H)$  are both well-defined then  $l(G + H)$  is defined and  $l(G + H) = l(G) + l(H)$ . Similarly, when right-length is defined for  $G$  and  $H$ , we have  $r(G + H) = r(G) + r(H)$ .

### 3 Integers and other dead ends

Let  $\mathbf{n}$  denote the game  $\{\mathbf{n} - 1 \mid \cdot\}$ , where  $\mathbf{0} = 0 = \{\cdot \mid \cdot\}$ . That is,  $\mathbf{n}$  is the position with the same game tree as the integer  $n$  in normal-play canonical form. In this paper the term ‘integer’ will always refer to such a position. Note that we should distinguish between the game  $\mathbf{n}$  and the number  $n$ , since, among other shortcomings,  $\mathbf{n} \not\geq \mathbf{n} - 1$  in general misère play. One property that does hold in both normal and misère play is that the disjunctive sum of positive integers  $\mathbf{n}$  and  $\mathbf{m}$  is the integer  $\mathbf{n} + \mathbf{m}$ , although this is not generally true (in misère games) if one of  $n$  or  $m$  is negative and the other positive. In this section we prove that the restricted universe of integers under misère play has much of the structure enjoyed by normal-play integers.

An integer is an example of a dead end: if  $n > 0$  then Right has no move in  $\mathbf{n}$  and no move in any follower of  $\mathbf{n}$ . Similarly, if  $n < 0$  then  $\mathbf{n}$  is a dead left end. Thus, the following results for ends in the dead-ending universe are true for all integers, modulo  $\mathcal{E}$ .

Our first result shows that when all games in a sum are dead ends, the outcome is completely determined by the left- and right-lengths of the games.

**Lemma 5.** *If  $G$  is a dead right end and  $H$  is a dead left end, then*

$$o^-(G + H) = \begin{cases} \mathcal{N}^- & \text{if } l(G) = r(H), \\ \mathcal{L}^- & \text{if } l(G) < r(H), \\ \mathcal{R}^- & \text{if } l(G) > r(H). \end{cases}$$

*Proof.* Each player has no choice but to play in their own game, and so the winner will be the player who can run out of moves first.  $\square$

We use Lemma 5 to prove the following theorem, which demonstrates the invertibility of all ends in  $\mathcal{E}$ . In particular, this shows that every integer has an additive inverse modulo  $\mathcal{E}$ .

**Theorem 6.** *If  $G$  is a dead end then  $G + \overline{G} \equiv 0 \pmod{\mathcal{E}}$ .*

*Proof.* Assume without loss of generality that  $G \neq 0$  is a dead right end. Since every follower of a dead end is also a dead end, Lemma 4 applies, with  $S$  the set of all dead left and right ends. It therefore suffices to show  $G + \overline{G} + X \in \mathcal{L}^- \cup \mathcal{N}^-$  for any left end  $X$  in  $\mathcal{E}$ . We have  $l(G) = r(\overline{G})$  and  $r(X) \geq 0$ , so  $l(G) \leq r(\overline{G}) + r(X) = r(\overline{G} + X)$ , which gives  $G + \overline{G} + X \in \mathcal{L}^- \cup \mathcal{N}^-$  by Lemma 5.  $\square$

**Corollary 7.** *If  $n$  is an integer then  $\mathbf{n} + \overline{\mathbf{n}} \equiv 0 \pmod{\mathcal{E}}$ .*

Note that equivalency in  $\mathcal{E}$  implies equivalency in all subuniverses of  $\mathcal{E}$ ; thus in the universe of integers alone, every game has an inverse.

Lemma 5 shows that when playing a sum of dead ends, both players aim to exhaust their own options as quickly as possible. This suggests that options with longer paths to zero will be dominated by shorter paths; in particular, we have that integers are totally ordered among dead ends, as established in Theorem 8 below. Note that this ordering only holds in the subuniverse of the closure<sup>4</sup> of dead ends, and not in the whole universe  $\mathcal{E}$ . In fact, we see immediately in Theorem 9 that distinct integers are pairwise incomparable modulo  $\mathcal{E}$ , just as they are in the general misère universe.

In the following arguments we frequently use the fact that, when  $H \in \mathcal{U}$  has an additive inverse modulo  $\mathcal{U}$ ,  $G \not\geq H \pmod{\mathcal{U}}$  if and only if  $G + \overline{H} \geq 0$ .

**Theorem 8.** *If  $n < m \in \mathbb{Z}$  then  $\mathbf{n} \not\geq \mathbf{m}$  modulo the closure of dead ends.*

*Proof.* By Corollary 7, it suffices to show  $\mathbf{n} + \overline{\mathbf{m}} \not\geq 0$  (equivalently,  $\mathbf{k} \not\geq 0$  for any negative integer  $k$ ), modulo the closure of dead ends. Let  $X$  be any game in the closure of dead ends; then  $X = Y + Z$  where  $Y$  is a dead right end and  $Z$  is a dead left end. Suppose Left wins  $X$  playing first; then by Lemma 5,  $l(Y) \leq r(Z)$ . We need to show Left wins  $\mathbf{k} + X$ , so that  $o^-(\mathbf{k} + X) \geq o^-(X)$ . Since  $\mathbf{k}$  is a negative integer,  $r(\mathbf{k})$  is defined and  $r(\mathbf{k}) = -k > 0$ . Thus  $l(Y) \leq r(Z) < r(Z) + r(\mathbf{k}) = r(Z + \mathbf{k})$ , which gives  $\mathbf{k} + Y + Z = \mathbf{k} + X \in \mathcal{L}^- \cup \mathcal{N}^-$ , by Lemma 5.  $\square$

In general,  $G \geq H$  under misère play implies  $G \geq H$  under normal play [10]; Theorem 8 shows this is not always the case for misère inequality modulo a restricted universe.

**Theorem 9.** *If  $n \neq m \in \mathbb{Z}$  then  $\mathbf{n} \parallel \mathbf{m} \pmod{\mathcal{E}}$ .*

*Proof.* Assume  $n > m$ . Then we have  $\mathbf{n} \not\geq \mathbf{m} \pmod{\mathcal{E}}$ , because  $\mathbf{n} + \overline{\mathbf{m}} \in \mathcal{R}^-$  while  $\mathbf{m} + \overline{\mathbf{m}} \equiv 0 \in \mathcal{N}^-$ . It remains to show  $\mathbf{n} \not\leq \mathbf{m}$ .

Define a family of games  $\lambda_k$  by

$$\lambda_1 = \{0 \mid -1\}, \lambda_k = \{0 \mid \lambda_{k-1}\}.$$

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<sup>4</sup>Since the sum of a dead left end and a dead right end may not be a dead end (or any end at all), the set of dead ends is not closed under disjunctive sum; thus the universe we consider is the *closure* of dead ends, as defined in Section 1.1.



Note that  $\mathbf{n} + \lambda_n \in \mathcal{L}^-$ , since Left wins playing first or second by ignoring  $\lambda_n$  and forcing Right to play there, bringing the game to  $-1$  with either Left or Right to play next.

If  $n > m \geq 0$  then  $\mathbf{m} + \lambda_n$  is in  $\mathcal{P}^-$  or  $\mathcal{R}^-$ : Left loses as soon as she plays in  $\lambda_n$ , and so plays only in  $\mathbf{m}$ , but (moving first) she will run out of moves in  $\mathbf{m}$  before  $\lambda_n$  is brought to  $-1$ . Thus  $\mathbf{n} \not\leq \mathbf{m}$  in this case, since Left can win  $\mathbf{n} + \lambda_n$  but not  $\mathbf{m} + \lambda_n$ .

If  $m < 0$  then let  $k = -m - 1$  and take  $X = \mathbf{k} + \lambda_{n+k}$ . As above,  $\mathbf{n} + \mathbf{k} + \lambda_{n+k} \in \mathcal{L}^-$ . However,  $\mathbf{m} + \mathbf{k} + \lambda_{n+k} \equiv -1 + \lambda_{n+k} \in \mathcal{N}^-$  since each player can move to a position from which the opponent is forced to move to zero. In this situation we see Left prefers  $\mathbf{n}$  over  $\mathbf{m}$ , so again  $\mathbf{n} \not\leq \mathbf{m}$ .  $\square$

Theorem 9 tells us that, modulo  $\mathcal{E}$ , the games  $\{0, \mathbf{1} \mid \cdot\}$  and  $\{0 \mid \cdot\}$  are distinguishable, as the option to 0 does not in general dominate the option to  $\mathbf{1}$ . Thus, in the dead-ending universe, there exist ends that are not integers. However, if we restrict ourselves to the subuniverse of dead ends, then the ordering given in Theorem 8 implies that every end reduces to an integer. This fact is presented in the following lemma.

**Lemma 10.** *If  $G$  is a dead end then  $G \equiv \mathbf{n}$ , modulo the closure of dead ends, where  $n = l(G)$  if  $G$  is a right end and  $n = -r(G)$  if  $G$  is a left end.*

*Proof.* Let  $G$  be a dead right end (the argument for left ends is symmetric). Assume by induction that every option  $G_i^L$  of  $G$  (necessarily a dead right end) is equivalent to the integer  $l(G_i^L)$ . Modulo dead ends, by Theorem 8, these left options are totally ordered; thus  $G = \{G_1^L \mid \cdot\}$  for  $G_1^L$  with smallest left-length. Then  $G$  is the canonical form of the integer  $l(G_1^L) + 1 = l(G)$ .  $\square$

Lemma 10 shows that the closure of dead ends has precisely the same monoid as the set of canonical-form integers. The game of DOMINEERING on  $1 \times n$  and  $n \times 1$  strips is an instance of these universes. The results of this section allow us to completely describe the monoid, which we present in Theorem 11.

**Theorem 11.** *Under the mapping  $\mathbf{n} \mapsto \alpha^n$ , the misère monoid of the set of normal-play canonical-form integers is*

$$\mathcal{M}_{\mathbb{Z}} = \langle 1, \alpha, \alpha^{-1} \mid \alpha \cdot \alpha^{-1} = 1 \rangle \cong (\mathbb{Z}, +),$$

*with outcome partition*

$$\mathcal{N}^- = \{0\}, \mathcal{L}^- = \{\alpha^{-n} \mid n \in \mathbb{N}\}, \mathcal{R}^- = \{\alpha^n \mid n \in \mathbb{N}\},$$

and total ordering

$$\alpha^n \not\geq \alpha^m \Leftrightarrow n < m.$$

## 4 Numbers

### 4.1 The monoid of $\mathbb{Q}_2$ .

We say that a game  $\mathbf{a}$  is a *non-integer number* in a universe  $\mathcal{U}$  if it is equivalent, modulo  $\mathcal{U}$ , to the normal-play canonical form of a (non-integer) dyadic rational:

$$\mathbf{a} = \frac{\mathbf{m}}{2^j} = \left\{ \frac{\mathbf{m} - 1}{2^j} \left| \frac{\mathbf{m} + 1}{2^j} \right. \right\},$$

with  $j > 0$  and  $m$  odd. The set of all integer and non-integer (combinatorial game) numbers is thus the set of *dyadic rationals*, which we denote by  $\mathbb{Q}_2$ . As we did for integers in the previous section, we now determine the outcome of a general sum of dyadic rationals and thereby describe the misère monoid of the closure of numbers.

Note that the sum of two non-integer numbers (even if both are positive) is not necessarily another number. For example, in general misère play,  $\mathbf{1} + \mathbf{1}/2 = \{\mathbf{1}/2, \mathbf{1} \mid \mathbf{2}\} \neq \mathbf{3}/2$  implies that  $\mathbf{1}/2 + \mathbf{1}/2 = \{\mathbf{1}/2 \mid \mathbf{1} + \mathbf{1}/2\} \neq \mathbf{1}$ . We will see that, unlike integers, the set of dyadic rationals is not closed under disjunctive sum even when restricted to the dead-ending universe; however, closure does occur when we restrict to numbers alone.

Lemma 14 below — analogous to Lemma 5 of the previous section — shows that the outcome of a sum of numbers is determined by the left- and right-lengths of the individual numbers. To prove this, we require Lemma 13, which establishes a relationship between the left- or right-lengths of numbers and their options; and to prove Lemma 13, we need the following proposition.

**Proposition 12.** *If  $a \in \mathbb{Q}_2 \setminus \mathbb{Z}$  then at least one of  $\mathbf{a}^{RL}$  and  $\mathbf{a}^{LR}$  exists, and either  $\mathbf{a}^L = \mathbf{a}^{RL}$  or  $\mathbf{a}^R = \mathbf{a}^{LR}$ .*

*Proof.* Let  $\mathbf{a} = \mathbf{m}/2^j$  with  $j > 0$  and  $m$  odd. If  $m \equiv 1 \pmod{4}$  then

$$\mathbf{a}^L = \frac{\mathbf{m} - 1}{2^j}, \mathbf{a}^R = \frac{\mathbf{m} + 1}{2^j} = \frac{\frac{\mathbf{m}+1}{2}}{2^{j-1}} = \left\{ \frac{\frac{\mathbf{m}-1}{2}}{2^{j-1}} \left| \frac{\frac{\mathbf{m}+3}{4}}{2^{j-1}} \right. \right\},$$

so  $\mathbf{a}^L = \mathbf{a}^{RL}$ . Otherwise,  $m \equiv 3 \pmod{4}$  and then

$$\mathbf{a}^L = \frac{\mathbf{m} - 1}{2^j} = \frac{\frac{\mathbf{m}-1}{2}}{2^{j-1}} = \left\{ \frac{\frac{\mathbf{m}-3}{2}}{2^{j-1}} \left| \frac{\frac{\mathbf{m}+1}{2}}{2^{j-1}} \right. \right\}, \mathbf{a}^R = \frac{\mathbf{m} + 1}{2^j},$$

so  $\mathbf{a}^R = \mathbf{a}^{LR}$ . □

Note that if  $a > 0$  is a dyadic rational then  $l(\mathbf{a}) = 1 + l(\mathbf{a}^L)$ , and if  $a < 0$  is a dyadic rational then  $r(\mathbf{a}) = 1 + r(\mathbf{a}^R)$ . We also have the following inequalities for left-lengths of right options and right-lengths of left options, when  $a$  is a non-integer dyadic rational.

**Lemma 13.** *If  $a \in \mathbb{Q}_2 \setminus \mathbb{Z}$  is positive then  $l(\mathbf{a}^R) \leq l(\mathbf{a})$ ; if  $a$  is negative then  $r(\mathbf{a}^L) \leq r(\mathbf{a})$ .*

*Proof.* Assume  $a > 0$  (the argument for  $a < 0$  is symmetric). Since  $\mathbf{a}$  is in canonical form, both  $\mathbf{a}^L$  and  $\mathbf{a}^R$  are positive numbers. If  $\mathbf{a}^L = \mathbf{a}^{RL}$  then  $l(\mathbf{a}^R) = 1 + l(\mathbf{a}^{RL}) = 1 + l(\mathbf{a}^L) = l(\mathbf{a})$ . Otherwise  $\mathbf{a}^R = \mathbf{a}^{LR}$ , by Proposition 12; then  $\mathbf{a}^L$  is not an integer because  $\mathbf{a}^{LR}$  exists, so by induction we obtain  $l(\mathbf{a}^R) = l(\mathbf{a}^{LR}) \leq l(\mathbf{a}^L) = l(\mathbf{a}) - 1 < l(\mathbf{a})$ . □

We can now determine the outcome of a general sum of numbers, both integer and non-integer.

**Lemma 14.** *If  $\{a_i\}_{1 \leq i \leq n}$  and  $\{b_i\}_{1 \leq i \leq m}$  are sets of positive and negative numbers, respectively, with  $k = \sum_{i=1}^n l(\mathbf{a}_i) - \sum_{i=1}^m r(\mathbf{b}_i)$ , then*

$$o^-\left(\sum_{i=1}^n \mathbf{a}_i + \sum_{i=1}^m \mathbf{b}_i\right) = \begin{cases} \mathcal{L}^- & \text{if } k < 0 \\ \mathcal{N}^- & \text{if } k = 0 \\ \mathcal{R}^- & \text{if } k > 0. \end{cases}$$

*Proof.* Let  $G = \sum_{i=1}^n \mathbf{a}_i + \sum_{i=1}^m \mathbf{b}_i$ . All followers of  $G$  are also of this form, so assume the result holds for every proper follower of  $G$ . Suppose  $k < 0$ . If  $n = 0$  then Left will run out of moves first because Left cannot move last in any negative number. So assume  $n > 0$ . Left moving first can move in an  $\mathbf{a}_i$  to reduce  $k$  by one (since  $l(\mathbf{a}_i^L) = l(\mathbf{a}_i) - 1$ ), which is a left-win position by induction. If Right moves first in an  $\mathbf{a}_i$  then  $k$  does not increase, since  $l(\mathbf{a}_i^R) \leq l(\mathbf{a}_i)$  by Lemma 13, so the position is a left-win by induction; if Right moves first in a  $\mathbf{b}_i$  then  $k$  does increase by one, but Left can respond in an  $\mathbf{a}_i$  (since  $n > 0$ ) to bring  $k$  down again, leaving another left-win position, by induction. Thus  $G \in \mathcal{L}^-$  if  $k < 0$ .

The argument for  $k > 0$  is symmetric. If  $k = 0$  then either  $G = 0$  is trivially next-win, or both  $n$  and  $m$  are at least 1 and both players have a good first move to change  $k$  in their favour. □

Lemma 14 shows that in general misère play, the outcome of a sum of numbers is completely determined by the left-lengths and right-lengths of the positive and negative components, respectively. From this we can conclude that, modulo the closure of canonical-form numbers, a positive number  $\mathbf{a}$  is equivalent to every other number with left-length  $l(\mathbf{a})$ . In particular, every positive number  $\mathbf{a}$  is equivalent to the integer  $\mathbf{l}(\mathbf{a})$ . This is Corollary 15 below; together with Theorem 18, it will allow us to describe the monoid of canonical-form numbers.

**Corollary 15.** *If  $\mathbf{a}$  is a number, then*

$$\mathbf{a} \equiv \begin{cases} \mathbf{l}(\mathbf{a}) & \text{if } a \geq 0, \\ -\mathbf{r}(\mathbf{a}) & \text{if } a < 0. \end{cases}$$

As an example, the dyadic rational  $\mathbf{1}/\mathbf{2}$  is equivalent to  $\mathbf{l}(\mathbf{1}/\mathbf{2}) = \mathbf{1}$ , and  $-\mathbf{3}/\mathbf{4} \equiv -\mathbf{r}(-\mathbf{3}/\mathbf{4}) = -\mathbf{2}$ , modulo  $\mathbb{Q}_2$ . Note that these equivalencies do not hold in the larger universe of  $\mathcal{E}$ ; indeed, as we see in section 4.2, if  $a \neq b$  are numbers then  $a \not\equiv b \pmod{\mathcal{E}}$ .

We see then that the closure of numbers is isomorphic to the closure of just integers; when restricted to numbers alone, every non-integer is equivalent to an integer. Thus the misère monoid of numbers, given below, is the same monoid presented in Theorem 11. The partial order of the set of numbers, modulo  $\mathcal{E}$ , is described in Section 4.2.

**Theorem 16.** *Under the mapping  $\mathbf{a} \mapsto \alpha^n$ , where  $n = l(G)$  if  $a \geq 0$  and  $n = r(G)$  if  $a < 0$ , the misère monoid of the set of canonical-form dyadic rationals is*

$$\mathcal{M}_{\mathbb{Q}_2} = \langle 1, \alpha, \alpha^{-1} \mid \alpha \cdot \alpha^{-1} = 1 \rangle \cong (\mathbb{Z}, +),$$

*with outcome partition*

$$\mathcal{N}^- = \{0\}, \mathcal{L}^- = \{\alpha^{-n} \mid n \in \mathbb{N}\}, \mathcal{R}^- = \{\alpha^n \mid n \in \mathbb{N}\}.$$

As with integers, some of the structure found in the number universe is also present in the larger universe  $\mathcal{E}$ . We end this subsection with a proof that all numbers — not just integers — are invertible in the universe of dead-ending games. We require the following lemma, an extension of Lemma 14.

**Lemma 17.** *If  $\{a_i\}_{1 \leq i \leq n}$  and  $\{b_i\}_{1 \leq i \leq m}$  are sets of positive and negative numbers, respectively, and  $\sum_{i=1}^n l(\mathbf{a}_i) - \sum_{i=1}^m r(\mathbf{b}_i) < 0$ , then*

$$o^-\left(\sum_{i=1}^n \mathbf{a}_i + \sum_{i=1}^m \mathbf{b}_i + X\right) = \mathcal{L}^-,$$

for any left end  $X \in \mathcal{E}$ .

*Proof.* The argument from Theorem 14 works again, since if Right uses his turn to play in  $X$  then Left responds with a move in  $\mathbf{a}_1$  to decrease  $k$  by 1, which is a win for Left by induction.  $\square$

**Theorem 18.** *If  $a \in \mathbb{Q}_2$  then  $\mathbf{a} + \bar{\mathbf{a}} \equiv 0 \pmod{\mathcal{E}}$ .*

*Proof.* Without loss of generality we can assume  $a$  is positive. Since every follower of a number is also a number, we can use Lemma 4. That is, it suffices to show  $\mathbf{a} + \bar{\mathbf{a}} + X \in \mathcal{L}^- \cup \mathcal{N}^-$  for any left end  $X \in \mathcal{E}$ . If  $X = 0$  this is true by Lemma 14. If  $X \neq 0$  then we claim  $\mathbf{a} + \bar{\mathbf{a}} + X \in \mathcal{L}^-$ ; assume this holds for all followers of  $\mathbf{a}$ . Left can win playing first on  $\mathbf{a} + \bar{\mathbf{a}} + X$  by moving to  $\mathbf{a}^L$ , since  $l(\mathbf{a}^L) - r(\bar{\mathbf{a}}) = l(\mathbf{a}^L) - l(\mathbf{a}) < 0$  implies  $\mathbf{a}^L + \bar{\mathbf{a}} + X \in \mathcal{L}^-$  by Lemma 17. If Right plays first in  $X$  then again Left wins by moving  $\mathbf{a}$  to  $\mathbf{a}^L$ ; if Right plays first in  $\bar{\mathbf{a}}$  then Left copies in  $\mathbf{a}$  and wins on  $\mathbf{a}^L + \bar{\mathbf{a}}^L + X \in \mathcal{L}^-$  by induction.  $\square$

Theorem 18 shows that in dead-ending games like DOMINEERING, HACKENBUSH, etc., any position corresponding to a normal-play canonical-form number has an additive inverse under misère play. So, for example, the positions in Figure 2 would cancel each other in a game of misère HACKENBUSH.

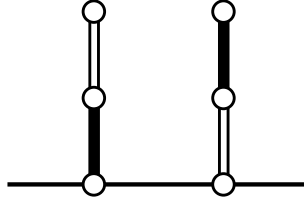


Figure 2: Normal-play canonical forms of  $1/2$  and  $-1/2$  in HACKENBUSH.

## 4.2 The partial order of numbers inside $\mathcal{E}$ .

In section 3 we found that all integers were incomparable in the dead-ending universe. We will see now that non-integer numbers are a bit more cooperative; although not totally ordered, we do have a nice characterization of the

partial order of numbers in the universe  $\mathcal{E}$ . First note that any two distinct numbers are distinguishable modulo  $\mathcal{E}$ ; this is an immediate corollary of the following theorem of [4], which extends a result of [10] referenced earlier.

**Theorem 19.** [4] *If  $G \geq H \pmod{\mathcal{E}}$  then  $G \geq H$  in normal play.*

**Corollary 20.** *If  $a, b \in \mathbb{Q}_2$  and  $a \neq b$  then  $\mathbf{a} \neq \mathbf{b} \pmod{\mathcal{E}}$ .*

Theorem 19 says that if  $\mathbf{a} \geq \mathbf{b} \pmod{\mathcal{E}}$  then  $a \geq b$  as real numbers (or as normal-play games). The converse is clearly not true for integers, by Theorem 9; it is also not true for non-integers, since  $1/2 - 1/2 \in \mathcal{N}^-$  while  $3/4 - 1/2 \in \mathcal{R}^-$  (which the reader can verify), so that  $1/2 \not\geq 3/4 \pmod{\mathcal{E}}$ . Theorem 23 shows that the additional stipulation  $l(\mathbf{a}) \leq l(\mathbf{b})$  is sufficient for  $\mathbf{a} \geq \mathbf{b} \pmod{\mathcal{E}}$ . To prove this result we need the following lemmas. As before, non-bolded symbols represent actual numbers, so that ‘ $a < b$ ’ indicates inequality of  $a$  and  $b$  as rational numbers (or as normal-play games), and  $a^L$  means the rational number corresponding to the left-option of the game  $\mathbf{a}$  in canonical form. Recall that if  $\mathbf{x} = \{\mathbf{x}^L | \mathbf{x}^R\}$  is in (normal-play) canonical form then  $\mathbf{x}$  is the *simplest* number (i.e., the number with smallest birthday) such that  $x^L < x < x^R$ . Thus, if  $x^L < x, y < x^R$  and  $x \neq y$  then  $\mathbf{x}$  is simpler than  $\mathbf{y}$ .

**Lemma 21.** *If  $a$  and  $b$  are positive numbers such that  $a^L < b < a$ , then  $l(\mathbf{a}^L) < l(\mathbf{b})$ .*

*Proof.* We have  $a^L < b < a < a^R$ , so  $\mathbf{a}$  must be simpler than  $\mathbf{b}$ . Thus  $b^L \geq a^L$ , since otherwise  $b^L < a^L < b < b^R$  would imply that  $\mathbf{b}$  is simpler than  $\mathbf{a}^L$ , which is simpler than  $\mathbf{a}$ . Now, if  $b^L = a^L$  then  $l(\mathbf{a}^L) = l(\mathbf{b}^L) = l(\mathbf{b}) - 1 < l(\mathbf{b})$ , and if  $b^L > a^L$  then by induction  $a^L < b^L < b < a$  gives  $l(\mathbf{a}^L) < l(\mathbf{b}^L) = l(\mathbf{b}) - 1 < l(\mathbf{b})$ .  $\square$

Lemma 21 is used to prove Lemma 22 below, which is needed for the proof of Theorem 23. Note that in the following two arguments we frequently use the fact that, if  $a \geq b \pmod{\mathcal{E}}$ , then Left wins on the position  $a + \bar{b} + X$  whenever she wins  $X \in \mathcal{E}$ .

**Lemma 22.** *If  $a$  and  $b$  are positive numbers such that  $a^L < b < a$ , then  $\mathbf{a} \not\geq \mathbf{b} \pmod{\mathcal{E}}$ .*

*Proof.* Note that  $b \notin \mathbb{Z}$  since there are no integers between  $a^L$  and  $a$  if  $\mathbf{a}$  is in canonical form. We must show that Left wins  $\mathbf{a} + \bar{\mathbf{b}} + X$  whenever she wins  $X \in \mathcal{E}$ .

*Case 1:  $b^R = a$ .*

Left can win  $\mathbf{a} + \bar{\mathbf{b}} + X$  by playing her winning strategy on  $X$ . If Right moves in  $\mathbf{a} + \bar{\mathbf{b}}$  to  $\mathbf{a}^R + \bar{\mathbf{b}} + X'$ , then Left responds to  $\mathbf{a}^R + \bar{\mathbf{b}}^R + X' = \mathbf{a}^R + \bar{\mathbf{a}} + X'$ , which she wins by induction since  $a^{RL} \leq a^L$  (see Proposition 12) gives  $a^{RL} < a < a^R$ . If Right moves to  $\mathbf{a} + \bar{\mathbf{b}}^R + X' = \mathbf{b}^R + \bar{\mathbf{b}}^R + X'$ , with  $X' \in \mathcal{L}^- \cup \mathcal{P}^-$  (since Left is playing her winning strategy in  $X$ ), then Left's response depends on whether  $\mathbf{b}^{RL} = \mathbf{b}^L$  or  $\mathbf{b}^{LR} = \mathbf{b}^R$ : if the former, Left moves to  $\mathbf{b}^{RL} + \bar{\mathbf{b}}^R + X' = \mathbf{b}^L + \bar{\mathbf{b}}^L + X' \equiv X' \pmod{\mathcal{E}}$ ; if the latter then Left moves to  $\mathbf{b}^R + \bar{\mathbf{b}}^{LR} + X' = \mathbf{b}^R + \bar{\mathbf{b}}^R + X' \equiv X'$ . In either case Left wins as the previous player on  $X' \in \mathcal{L}^- \cup \mathcal{P}^-$ .

When Left runs out of moves in  $X$ , she moves to  $\mathbf{a}^L + \bar{\mathbf{b}} + X$ . By Lemma 21 we know  $l(\mathbf{a}^L) < l(\mathbf{b})$ , and this gives  $\mathbf{a}^L + \bar{\mathbf{b}} + X \in \mathcal{L}^-$  by Lemma 17.

*Case 2:  $b^R \neq a$ .*

Note that  $b^R$  cannot be greater than  $a$ , since  $a^L < b < a < a^R$  implies  $\mathbf{a}$  is simpler than  $\mathbf{b}$ , while  $b^L < b < a < b^R$  would imply that  $\mathbf{b}$  is simpler than  $\mathbf{a}$ . So  $b^R < a$ , and together with  $a^L < b < b^R$  this gives  $a^L < b^R < a$ , which shows  $\mathbf{a} \geq \mathbf{b}^R \pmod{\mathcal{E}}$  by induction. Similarly  $b^{RL} \leq b^L < b < b^R$  implies  $\mathbf{b}^R \geq \mathbf{b} \pmod{\mathcal{E}}$ , by Case 1. Then by transitivity we have  $\mathbf{a} \geq \mathbf{b} \pmod{\mathcal{E}}$ .  $\square$

With Lemma 22 we can now prove Theorem 23 below. The symmetric result for negative numbers also holds.

**Theorem 23.** *If  $a$  and  $b$  are positive numbers such that  $a > b$  and  $l(\mathbf{a}) \leq l(\mathbf{b})$ , then  $\mathbf{a} \geq \mathbf{b} \pmod{\mathcal{E}}$ .*

*Proof.* By Corollary 20 we have  $\mathbf{a} \not\equiv \mathbf{b} \pmod{\mathcal{E}}$ , and so it suffices to show  $\mathbf{a} \geq \mathbf{b} \pmod{\mathcal{E}}$ . Again we have  $b \notin \mathbb{Z}$ . Since  $a > b$ , if  $b > a^L$  then Lemma 22 gives  $\mathbf{a} \geq \mathbf{b} \pmod{\mathcal{E}}$  as required. So assume  $b \leq a^L$ . Again let  $X \in \mathcal{E}$  be a game which Left wins playing first; we must show Left wins  $\mathbf{a} + \bar{\mathbf{b}} + X$  playing first. Left should follow her winning strategy from  $X$ . If Right plays to  $\mathbf{a} + \bar{\mathbf{b}}^L + X'$ , where  $X' \in \mathcal{L}^- \cup \mathcal{P}^-$ , then Left responds with  $\mathbf{a}^L + \bar{\mathbf{b}}^L + X'$ , which

she wins by induction:  $b^L < b \leq a^L$  and  $l(\mathbf{b}^L) = l(\mathbf{b}) - 1 \geq l(\mathbf{a}) - 1 = l(\mathbf{a}^L)$  implies  $\mathbf{a}^L \geq \mathbf{b}^L \pmod{\mathcal{E}}$ .

If Right plays to  $\mathbf{a}^R + \bar{\mathbf{b}} + X'$  (assuming this move exists — that is, assuming  $a \notin \mathbb{Z}$ ) then Left's response is  $\mathbf{a}^{RL} + \bar{\mathbf{b}} + X'$ , if  $a^{RL} > b$ , or  $\mathbf{a}^R + \bar{\mathbf{b}}^R + X'$  if  $a^{RL} \leq b$ . In the first case Left wins by induction because  $a^{RL} > b$  and  $l(\mathbf{a}^{RL}) = l(\mathbf{a}^R) - 1 \leq l(\mathbf{a}) - 1 < l(\mathbf{b})$  implies  $\mathbf{a}^{RL} \geq \mathbf{b} \pmod{\mathcal{E}}$ . In the latter case, note firstly that in fact  $a^{RL} \neq b$ , since we have already seen that as games they have different left-lengths. Then we see  $a^{RL} < b < a < a^R < a^{RR}$ , which shows  $\mathbf{a}^R$  must be simpler than  $\mathbf{b}$ . This gives  $b^R \leq a^R$ , as otherwise  $b^L < b < a < a^r < b^R$  would imply that  $b$  is simpler than  $a^R$ . We can now apply Lemma 22 to conclude that  $\mathbf{a}^R \geq \mathbf{b}^R \pmod{\mathcal{E}}$ , and so Left wins  $\mathbf{a}^R + \bar{\mathbf{b}}^R + X'$ , with  $X' \in \mathcal{L}^- \cup \mathcal{P}^-$ , as the second player.

Finally, if Left runs out of moves in  $X$  then she moves to  $\mathbf{a}^L + \bar{\mathbf{b}} + X''$  where  $X''$  is a dead left end; then Left wins by Lemma 17 because  $l(\mathbf{a}^L) < l(\mathbf{a}) \leq l(\mathbf{b}) = r(\bar{\mathbf{b}})$ .  $\square$

**Corollary 24.** *For positive numbers  $a, b \in \mathbb{Q}_2$ ,  $\mathbf{a} \geq \mathbf{b} \pmod{\mathcal{E}}$  if and only if  $a > b$  and  $l(\mathbf{a}) \leq l(\mathbf{b})$ .*

*Proof.* We need only prove the converse of Theorem 23. Suppose  $a > b$  and  $l(\mathbf{a}) > l(\mathbf{b})$ ; then by [4] it cannot be that  $\mathbf{a} \leq \mathbf{b} \pmod{\mathcal{E}}$ , so we need only show  $\mathbf{a} \not\geq \mathbf{b} \pmod{\mathcal{E}}$ . We have  $\mathbf{b} + \bar{\mathbf{b}} \in \mathcal{N}^-$ , while  $\mathbf{a} + \bar{\mathbf{b}} \in \mathcal{R}^-$ , since in isolation the latter sum is equivalent to the positive integer  $l(\mathbf{a}) - l(\mathbf{b})$ , by Theorem 16. Thus  $\mathbf{a} \not\geq \mathbf{b} \pmod{\mathcal{E}}$ .  $\square$

To completely describe the partial order of numbers within  $\mathcal{E}$ , it remains to consider the comparability of  $\mathbf{a}$  and  $\mathbf{b}$  when  $a \geq 0$  and  $b < 0$  (or, symmetrically, when  $a > 0$  and  $b \leq 0$ ). As before we cannot have  $\mathbf{a} \leq \mathbf{b} \pmod{\mathcal{E}}$ , and the same argument as above ( $\mathbf{b} + \bar{\mathbf{b}} \in \mathcal{N}^-$  and  $\mathbf{a} + \bar{\mathbf{b}} \in \mathcal{R}^-$ ) shows  $\mathbf{a} \not\geq \mathbf{b} \pmod{\mathcal{E}}$ . The results of this section are summarized below.

**Theorem 25.** *The partial order of  $\mathbb{Q}_2$ , modulo  $\mathcal{E}$ , is given by*

$$\begin{aligned} \mathbf{a} &\equiv \mathbf{b} \pmod{\mathcal{E}} && \text{if } a = b; \\ \mathbf{a} &\geq \mathbf{b} \pmod{\mathcal{E}} && \text{if } 0 < a < b \text{ and } l(\mathbf{a}) \leq l(\mathbf{b}), \\ &&& \text{or } b < a < 0 \text{ and } r(\mathbf{b}) \leq r(\mathbf{a}); \\ \mathbf{a} &\parallel \mathbf{b} \pmod{\mathcal{E}} && \text{otherwise.} \end{aligned}$$



## 5 Zeros in the dead-ending universe

We have found that integer and non-integer numbers, as well as all ends, satisfy  $G + \overline{G} \equiv 0 \pmod{\mathcal{E}}$ . It is not the case that every game in  $\mathcal{E}$  has an additive inverse; for example,  $* + * \not\equiv 0 \pmod{\mathcal{E}}$ , although the equivalence does hold in the dicot universe  $\mathcal{D} \subset \mathcal{E}$ . Likewise, many familiar ‘all-small’ games from normal-play, which have inverses in the dicot universe<sup>5</sup>, are not invertible here.

The following lemma describes an infinite family of games that are not invertible in the universe of dead-ending games.

**Lemma 26.** *If  $G = \{\mathbf{n}_1, \dots, \mathbf{n}_k \mid \overline{\mathbf{n}}_1, \dots, \overline{\mathbf{n}}_k\}$  with each  $n_i \in \mathbb{N}$ , then  $G + \overline{G} \not\equiv 0 \pmod{\mathcal{E}}$ .*

*Proof.* Let  $X = \{\mathbf{n}_1, \dots, \mathbf{n}_k \mid \cdot\} \in \mathcal{R}^-$ . Note that  $G = \overline{G}$ . We describe a winning strategy for Left playing second in the game  $G + \overline{G} + X = G + G + X$ . Right has no first move in  $X$ , so Right’s move is of the form  $G + \overline{\mathbf{n}}_i + X$ . Left can respond by moving  $X$  to  $\mathbf{n}_i$ , leaving  $G + 0$ . Now Right plays in  $G$  to a nonpositive integer, which as a right end must be in  $\mathcal{L}^-$  or  $\mathcal{N}^-$ .  $\square$

We conclude with an infinite family of games that are equivalent to zero in the dead-ending universe, which are not all of the form  $G + \overline{G}$  for some  $G$ . These games are illustrated in Figure 3.

**Theorem 27.** *If  $G$  is a dead-ending game such that every  $G^L$  is a left end with an option to zero and every  $G^R$  is a right end with an option to zero, then  $G \equiv 0 \pmod{\mathcal{E}}$ .*

*Proof.* Let  $X$  be any game in  $\mathcal{E}$  and suppose Left wins  $X$ . Then Left wins  $G + X$  by following her strategy in  $X$ . If Right plays in  $G$  then she moves to  $G^R + X'$  from a position  $G + X'$  with  $X' \in \mathcal{L}^- \cup \mathcal{P}^-$ ; Left can respond to  $0 + X'$  and win as the second player. If both players ignore  $G$  then eventually Left runs out of moves in  $X$  and plays to  $G^L + X''$ , where  $X''$  is a left end. But  $G^L$  is also a left end, so the sum is a left-win by Lemma 3.  $\square$

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<sup>5</sup>The games  $\uparrow = \{0 \mid *\}$ ,  $\downarrow = \{*\mid 0\}$ , and all other day-2 misère dicots with the exception of  $*2 = \{0, * \mid 0, *\}$ , are shown to be invertible modulo  $\mathcal{D}$  in the first author’s thesis (in progress), using Lemma 4.

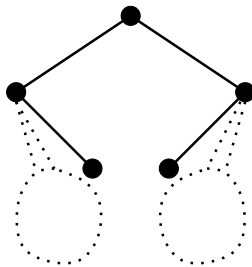


Figure 3: An infinite family of games equivalent to zero modulo  $\mathcal{E}$ .

## 6 Future directions

From this first initial investigation, the universe of dead-ending games appears to be filled with potential for successful misère analysis. It includes as subuniverses many of the games that have already excited interest among combinatorial game theorists; we hope some of the techniques of the present paper can be fruitfully applied to these subuniverses.

A natural extension of this work would include analysis of specific games, such as NOGO, COL, SNORT, etc., in the context of the dead-ending universe. It would also be interesting to consider other properties, besides dead-ending, of the ‘placement games’ described in our opening paragraph.

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## 8 References

### References

- [1] M.H. Albert, R.J. Nowakowski, D. Wolfe, Lessons in Play, A K Peters, Ltd., MA, 2007.

- [2] M.R. Allen, An investigation of misère partizan games, PhD thesis, Dalhousie University, 2009.
- [3] M.R. Allen, Peeking at partizan misère quotients, in: Games of No Chance 4, to appear.
- [4] P. Dorbec, G. Renault, A.N. Siegel, E. Sopena, Dicots, and a taxonomic ranking for misère games, unpublished manuscript, 2012.
- [5] N.A. McKay, R. Milley, R.J. Nowakowski, Misère-play hackenbush sprigs, preprint, 2012; available at arxiv 1202:5654.
- [6] G.A. Mesdal, P. Ottaway, Simplification of partizan games in misère play, INTEGERS: Electronic J. Comb. Number Theory 7 (2007) #G6.
- [7] R. Milley, R.J. Nowakowski, P. Ottaway, The misère monoid of one-handed alternating games, INTEGERS: Electronic J. Comb. Number Theory 12B (2012) #A1.
- [8] T.E. Plambeck, Taming the wild in impartial combinatorial games, INTEGERS: Electronic J. Comb. Number Theory, 5 (2005) #G5.
- [9] T.E. Plambeck, A.N. Siegel, Misère quotients for impartial games, J. Comb. Theory, Series A, 115(4) (2008) 593 – 622.
- [10] A.N. Siegel, Misère canonical forms of partizan games, preprint, 2012; available at arxiv math/0703565.